

# A Certain Class of Curie-Weiss Models

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## Abstract

By using a formal analogy between statistical mechanics of mean field spin systems and analytical mechanics of viscous liquids -at first pointed out by Francesco Guerra, then recently developed by the authors- we give the thermodynamic limit of the free energy and the critical behavior of Curie Weiss models for a certain class of generalized spin variables. Then, with the same techniques, we give a complete picture of the bipartite Curie-Weiss model, dealing with the same class of generalized spins. Ultimately we analyze further the existence of a minmax principle for the latter which mirrors the standard variational principle of canonical thermodynamics when generalized to multiple interacting parties.

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## 1 Introduction

The investigation of statistical mechanics of mean field spin systems is experiencing an increasing interest in the last decades. The motivations are two-fold: from one side, at the rigorous mathematical level, a clear picture is still to be achieved (it is enough to think at the whole community dealing with the case of random interactions as in glasses [12]), at the applied level, these toy models are starting to be used in several different context, ranging from quantitative sociology [3] to theoretical immunology [13].

It is then obvious the need for always stronger and simpler methods to analyze the enormous amount of "variations on theme", the theme being the standard and simplest dichotomic Curie-Weiss model (CW) [1].

Recently, inspired by the pioneering work of Francesco Guerra [7], we paved a clear way to manage models with self-averaging order parameters by using the CW prototype as a guide [6]. Here, at first, we apply our scheme to work out the single-party CW with general spins (i.e. continuous spins with compact support and symmetric probability measure [9]) and we solve in all details its thermodynamics. Then we switch to the case of bipartite systems with generalized spins [5], both offering a clear picture of the thermodynamics as well as a digression on the connection of the coupled self-consistent equations for these models with the existence of an underlying minmax principle [8].

Overall we covered both the ways of investigation: from one side our analysis is mathematically clear (no powerful but not fully rigorous methods as replica trick or saddle points are used),

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from the other side it is automatically ready for being implemented into applied scenarios, i.e. the generalized bipartite system extends competitions in decision making [11] among two communities by allowing their constituent to assume "softer" viewpoints among each other (with respect to the accept/reject perspective), being the spins ruling their will continuous instead of dichotomic.

## 2 The ferromagnet with continuously distributed spin variables

We deal with a system made by  $N$  i.i.d. spin r.v.  $\sigma_i$ ,  $i = 1 \dots N$ , with probability measure  $\mu(\sigma)$ , having the following properties:

*i)*  $\mu(\sigma) = \mu(-\sigma)$ ,  
*i.e.* it is symmetric;

*ii)*  $\exists L : \forall \epsilon > 0 \int_{L/2}^{L/2+\epsilon} d\mu(\sigma) f(\sigma) = \int_{-L/2-\epsilon}^{-L/2} d\mu(\sigma) f(\sigma) = 0$ ,  
*i.e.* it has compact support  $[-L/2, L/2]$ .

In particular, denoting with  $\mathbb{E}_{\sigma_N} = \int_{-\infty}^{+\infty} d\mu(\sigma_1) \dots d\mu(\sigma_N)$  the expectation values with respect to the  $N$  spin variables, we notice that from *i)* it follows that  $\mathbb{E}_{\sigma}[\sigma] = 0$ , and from *ii)* that, for a given bounded function of spin  $f(\sigma)$ , it has to be  $\mathbb{E}_{\sigma}[f(\sigma)] \leq L[\sup_{\sigma \in [-L/2, L/2]} f(\sigma)]$ .

The spins interact each other, in the way described by the Hamiltonian  $H_N(\sigma, h)$

$$H_N(\sigma, h) = -\frac{1}{N} \sum_{(i,j)}^{N,N} \sigma_i \sigma_j - h \sum_i^N \sigma_i. \quad (1)$$

Partition function, pressure and free energy per site are defined as usual as

$$\begin{aligned} Z_N(\beta, h) &= \mathbb{E}_{\sigma_N} e^{-\beta H_N(\sigma, h)}, \\ A_N(\beta, h) &= \frac{1}{N} \log Z_N(\beta, h), \\ f_N(\beta, h) &= -\frac{1}{\beta} A_N(\beta, h). \end{aligned}$$

Of course we are interested in calculating the value of the free energy in thermodynamic limit, *i.e.* for  $N \rightarrow \infty$ , for describing the thermodynamics of the model. We can also define Boltzmann states of our system for a generic function of the  $N$  spins  $g_N$ , as

$$\langle g_N(\sigma) \rangle = \frac{\mathbb{E}_{\sigma_N} g_N(\sigma) \exp(-\beta H_N(\sigma, h))}{Z_N(\beta, h)}. \quad (2)$$

It is useful to define also the following quantities:

$$m_N = \frac{1}{N} \sum_i^N \sigma_i, \quad (3)$$

$$a_N = \frac{1}{N} \sum_i^N \sigma_i^2, \quad (4)$$

respectively the magnetization of the system, and the self overlap of spin variables (of course we have trivially  $a_N = 1 \forall N$  in the case of dichotomic spin). We always have trivially  $\langle a_N \rangle \leq L^2$ .

We can even express the Hamiltonian (1) in terms of (3) and (4). It is

$$H_N(\sigma, h) = -N\left(\frac{1}{2}m_N^2 + hm_N\right) + \frac{1}{2}a_N.$$

This is the starting point of the next section.

## 2.1 The free energy in the thermodynamic limit

We will follow the approach described in [7][6]. With this purpose, let us introduce the function  $\varphi_N(x, t) : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$  defined as

$$\varphi_N(x, t) = -\frac{1}{N} \log \mathbb{E}_{\sigma_N} \exp \left( \frac{tN}{2} m_N^2 + xNm_N \right). \quad (5)$$

This function plays the role of the pressure (up to a sign) of a model defined with a *Boltzmannfaktor*  $\exp(tNm_N^2/2 + xNm_N)$  depending by the two other parameters  $(x, t)$  (it is just a matter of names, since both  $t$  and  $\beta$  as well as  $x$  and  $h$  have the same range of definition). Anyhow it is strictly related to the pressure of our model, as states the following

**Lemma 1.** *It is, uniformly in  $N$*

$$|\varphi_N(x, t) + A_N(x, t)| \leq O\left(\frac{1}{N}\right). \quad (6)$$

Furthermore we have that  $\varphi_N(0, x) = A_N(0, x)$  is Lipschitz-continuous  $\forall N$ .

*Proof.* Due to convexity of logarithm we get

$$\begin{aligned} |\varphi_N(x, t) + A_N(x, t)| &= \frac{1}{N} \left| \log \left[ \frac{\mathbb{E}_{\sigma_N} \exp \left( \frac{tN}{2} m_N^2 + xNm_N \right) e^{\frac{ta_N}{2}}}{\mathbb{E}_{\sigma_N} \exp \left( \frac{tN}{2} m_N^2 + xNm_N \right)} \right] \right| \\ &\leq \frac{1}{N} \log \mathbb{E}_{\sigma} e^{\frac{ta_N}{2}} \leq \frac{L^2 t}{2N}, \end{aligned}$$

and (6) is proven. Furthermore, trivially it is  $\varphi_N(0, x) = -A_N(0, x) = -\log \mathbb{E}_{\sigma}[e^{x\sigma}]$ . Again by a convexity argument, joint with the compactness of the support of the  $\sigma$ -distribution, it is

$$\begin{aligned} |\varphi_N(x, 0) - \varphi_N(x_0, 0)| &= \left| \log \frac{\mathbb{E}_{\sigma} e^{x\sigma}}{\mathbb{E}_{\sigma} e^{x_0\sigma}} \right| \\ &= \left| \log \frac{\mathbb{E}_{\sigma} e^{(x-x_0)\sigma} e^{x_0\sigma}}{\mathbb{E}_{\sigma} e^{x_0\sigma}} \right| \leq \log \mathbb{E}_{\sigma} e^{|x-x_0|\sigma} \leq L|x-x_0|, \end{aligned} \quad (7)$$

hence  $\varphi_N(x, 0)$  is Lipschitz-continuous.  $\square$

**Remark 1.** *We have that in the thermodynamic limit  $A(\beta, h) = -\varphi(x = h, t = \beta)$ .*

**Remark 2.** *From (6) it is easily seen that in thermodynamic limit the definitions of state (2) and the one built with Boltzmannfaktor  $\exp(tNm_N^2/2 + xNm_N)$  do coincide (of course replacing  $(\beta, h)$  with  $(t, x)$ ), thus we actually won't distinguish them in the following.*

The main idea, for solving the thermodynamics encoded into our Hamiltonian, is to relate the statistical mechanics system to an effective mechanical one, in which we naturally identify  $x$  with space coordinate,  $t$  with time, and the function  $\varphi_N(x, t)$  with the mechanical action. In order to do this we notice that  $\varphi_N(x, t)$  satisfies the differential problem [6]

$$\begin{cases} \partial_t \varphi_N(x, t) + \frac{1}{2}(\partial_x \varphi_N(x, t))^2 - \frac{1}{2N} \partial_x^2 \varphi_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi_N(x, 0) = -\log \mathbb{E}_\sigma e^{x\sigma} & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \quad (8)$$

This is a Hamilton-Jacobi equation with a vanishing dissipative term in the thermodynamic limit. Defined  $u_N(x, t) = \partial_x \varphi_N(x, t)$  the velocity field, we notice that it corresponds to magnetization of the finite size system in our parallelism [6]. We have that  $u_N(x, t)$  satisfies a Burger's equation again with a mollifier dissipative term:

$$\begin{cases} \partial_t u_N(x, t) + u_N(x, t) \partial_x u_N(x, t) - \frac{1}{2N} \partial_x^2 u_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ u_N(x, 0) = -\mathbb{E}_\sigma \sigma e^{x\sigma} / \mathbb{E}_\sigma e^{x\sigma} & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \quad (9)$$

Thus the problem of the existence and uniqueness of the thermodynamic limit is here translated into the convergence of the viscous mechanical problem to the free one. We can use a theorem, that resumes a number of results obtained by Peter Lax [10] and assures the existence of the solution for free problem:

**Theorem 1.** *For a general differential problem*

$$\begin{cases} \partial_t \varphi(x, t) + \frac{1}{2}(\partial_x \varphi(x, t))^2 = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi(x, 0) = h(x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (10)$$

and

$$\begin{cases} \partial_t u(x, t) + u(x, t) \partial_x u(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ u(x, 0) = g(x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (11)$$

where  $h(x)$  is Lipschitz-continuous, and  $g(x) = h'(x) \in \mathcal{L}^\infty$ , it does exist and it is unique the function  $y(x, t) : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$  such that

$$\varphi(x, t) = \min_y \left\{ \frac{t}{2} \left( \frac{x - y}{t} \right)^2 + h(y) \right\} = \frac{t}{2} \left( \frac{x - y(x, t)}{t} \right)^2 + h(y(x, t)) \quad (12)$$

is the unique weak solution of (10), and

$$u(x, t) = \frac{x - y(x, t)}{t} \quad (13)$$

is the unique weak solution of (11). Furthermore, the function  $x \rightarrow y(x, t)$  is not-decreasing.

It is easily seen that Lax's theorem gives us the solution for the free energy of the model. In fact, if we put  $u(x, t) = -M(x, t)$  in the solution of the free Burgers' equation, and use  $F$  as a short label standing for "free", we get that the minimizing function is  $y(x, t) = x + tM(x, t)$ , and the action of the mechanical model reads off as

$$\varphi_F(x, t) = -A(x, t) = \frac{t^2}{2} M^2(x, t) - \log \mathbb{E}_\sigma [\exp(\sigma(x + tM(x, t)))].$$

Therefore, we have the following expression for the free energy per site of our models,

$$\begin{aligned} f(\beta, h) &= \frac{1}{\beta} [\varphi_F(x, t)]_{(t=\beta, x=h)} \\ &= \frac{\beta}{2} M^2(h, \beta) - \frac{1}{\beta} \log \mathbb{E}_\sigma [\exp (\sigma(h + \beta M(h, \beta)))]. \end{aligned} \quad (14)$$

Finally we must prove convergence of the viscous problem to the free one. To this purpose, we can state the following

**Theorem 2.** *The function*

$$\varphi_N(x, t) = -\frac{1}{N} \log \sqrt{\frac{N}{t}} \int \frac{dy}{\sqrt{2\pi}} \exp [-N ((x - y)^2 / 2t - \log \mathbb{E}_\sigma \exp \sigma y)] \quad (15)$$

*does solve equation (8) and it is*

$$|\varphi_N(x, t) - \varphi_F(x, t)| \leq O\left(\frac{1}{N}\right).$$

*Furthermore, the function*

$$u_N(x, t) = -\frac{\int \frac{dy}{\sqrt{2\pi}} \frac{x-y}{t} \exp [-N ((x - y)^2 / 2t - \log \mathbb{E}_\sigma \exp \sigma y)]}{\int \frac{dy}{\sqrt{2\pi}} \exp [-N ((x - y)^2 / 2t - \log \mathbb{E}_\sigma \exp \sigma y)]} \quad (16)$$

*does solve equation (9) and it is*

$$|u_N(x, t) + M(x, t)| \leq O\left(\frac{1}{\sqrt{N}}\right).$$

The proof is exactly analogue to the one given in [6], and ultimately due to the uniform convexity of the exponent in (15) and (16), that we have here by construction, so we will not report it here.

Therefore we have proven the existence of the thermodynamic limit for free energy and magnetization of our model.

As symmetry breaking are fundamental even in statistical mechanics, we want to report hereafter some other considerations about the existence and the properties of a phase transition in our analogy.

## 2.2 Phase transition and shock waves

In this section we deeply study properties of the free Burgers' equation for the velocity field (11) (that we remind is the analogue of the magnetization). We can write the straight line trajectories of the free system (*i.e.* the system in thermodynamic limit):

$$\begin{cases} t &= s \\ x &= x_0 - s \mathbb{E}_\sigma \sigma e^{\sigma x_0} / \mathbb{E}_\sigma e^{\sigma x_0}. \end{cases} \quad (17)$$

As usual in these cases, we can find a solution for the magnetization along characteristics [4]. It is

$$-u(x, t) = M(x, t) = \frac{\mathbb{E}_\sigma \sigma \exp [\sigma(x + tM(x, t))]}{\mathbb{E}_\sigma \exp [\sigma(x + tM(x, t))]} \quad (18)$$

**Remark 3.** Putting  $(x = h, t = \beta)$  in (18) we recover the generalized self consistence equation for the magnetization. In particular, by choosing  $\mu(\sigma) = (1/2)[\delta(\sigma + 1) + \delta(\sigma - 1)]$ , we immediately recognize the well known hyperbolic tangent of the dichotomic CW model.

An important feature of the velocity field is that it is monotone with respect to  $x$ . Indeed it is

$$\partial_x u(x, t) = -\frac{A_{(x,y)}[\sigma^2]}{1 + tA_{(x,y)}[\sigma^2]} \leq 0,$$

since  $\forall(x, t)$

$$A_{(x,y)}[\sigma^2] = \frac{\mathbb{E}_\sigma \sigma^2 \exp[\sigma(x + tM(x, t))]}{\mathbb{E}_\sigma \exp[\sigma(x + tM(x, t))]} - \left( \frac{\mathbb{E}_\sigma \sigma \exp[\sigma(x + tM(x, t))]}{\mathbb{E}_\sigma \exp[\sigma(x + tM(x, t))]} \right)^2 \geq 0$$

This is known as the entropy condition for the velocity field in the theory of shock waves [4][10]. For  $M(x, t)$  it follows that

$$\partial_x M(x, t) \geq 0. \quad (19)$$

We have seen in [6] that in usual CW model, *i.e.* with dichotomic spin variables, the spontaneous symmetry breaking associated to the phase transition appears as a shock wave in our mechanical analogy. The same happens dealing with our generalized variables.

**Proposition 1.** The line  $(t > t_c, 0)$ , with  $t_c = \sup \frac{M_0}{x_0}$  is a shock wave for  $M(x, t)$ , and by putting  $M_\pm = \lim_{x \rightarrow 0^\pm} M(x, t)$ , it is  $M^+ = -M^-$ .

*Proof.* With a glance to characteristics (17) we notice that  $x = 0$  is a stable point of motion<sup>1</sup>. Furthermore for  $x = 0$  all the straight lines do intersect the  $x$ -axis in a certain time. Defining

$$t_c = \sup_{x_0} \frac{M(x_0, 0)}{x_0} = \sup_{x_0} \partial_x M(x_0, 0), \quad (20)$$

we have that the line  $(t > t_c, 0)$  is a discontinuity line for  $M(x, t)$  since every point on this line is an intersection point of characteristics, *i.e.* it is a shock waves for the velocity field  $u(x, t)$ . We notice from (20) that, since we have  $\inf(M^2(x, t)) = 0$ , it must be

$$t_c = \sup_{x_0} \frac{\mathbb{E}_\sigma \sigma^2 \exp[\sigma(x + tM(x, t))]}{\mathbb{E}_\sigma \exp[\sigma(x + tM(x, t))]} \leq L^2.$$

On the other hand, we have that for every time there certainly exists a neighbors of  $x = 0$  where the function  $M(x, t)$  is smooth. Thus we are allowed to use the Rankine-Hugoniot condition for the jump along discontinuity [4][10] for stating  $M_+^2 = M_-^2$ . This last result, coupled with (19), completes the proof.  $\square$

### 3 Bipartite models

We are now interested in considering a set of  $N$  spin variables, in which is precisely defined a partition in two subsets of size respectively  $N_1$  and  $N_2$ . We assume the variable's label of the first subset as  $\sigma_i$ ,  $i = 1, \dots, N_1$ , while the spins of the second one are introduced by  $\tau_j$ ,  $j = 1, \dots, N_2$ .

<sup>1</sup>It is actually due to the parity of probability measure of  $\sigma$ .

For each subset all the spins are i.i.d. r.v., with probability measure as discussed above, but in principle  $\mu(\sigma)$  could be different by  $\mu(\tau)$ . Of course we have  $N_1 + N_2 = N$ , and we name the relative size of the two subset  $N_2/N_1 = \alpha_N$ . To avoid a trivial behavior of the model, we assume that the thermodynamic limit is performed in such a way that  $\alpha = \lim_N \alpha_N$  is well defined. The spins interact via the Hamiltonian  $H_N(\sigma, \tau, h_1, h_2)$ :

$$H_N(\sigma, \tau, h_1, h_2) = -\frac{1}{N_1} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sigma_i \tau_j - h_1 \sum_{i=1}^{N_1} \sigma_i - h_2 \sum_{j=1}^{N_2} \tau_j.$$

We notice that spins in each subsystem interact only with spins in the other one, but not among themselves. Partition function, pressure and free energy per site for the model are defined naturally, in agreement with the previous section:

$$\begin{aligned} Z_N(\beta, h_1, h_2) &= \mathbb{E}_{\sigma_{N_1}} \mathbb{E}_{\tau_{N_2}} e^{-\beta H_N(\sigma, \tau, h_1, h_2)}, \\ A_N(\beta, h_1, h_2) &= \frac{1}{N_1} \log Z_N(\beta, h_1, h_2), \\ f_N(\beta, h_1, h_2) &= -\frac{1}{\beta} A_N(\beta, h_1, h_2). \end{aligned}$$

**Remark 4.** *It should be noticed that, for coherence with already known bipartite models (as i.e. the Hopfield model [2]), we choose  $N_1$ , instead of  $N$ , as the normalization factor inside the free energy density and pressure. As we are considering the extensive scaling among the two parties, i.e.  $N_2 = \alpha_N N_1$  and  $\lim_{N \rightarrow \infty} \alpha_N = \alpha \in \mathbb{R}^+$ , this simply shifts the overall result by a factor  $(1 + \alpha)^{-1}$ .*

We can also specify the Boltzmann states of our system as

$$\langle g_N(\sigma, \tau) \rangle = \frac{\mathbb{E}_{\sigma_{N_1}} \mathbb{E}_{\tau_{N_2}} g_N(\sigma, \tau) \exp(-\beta H_N(\sigma, \tau, h_1, h_2))}{Z_N(\beta, h_1, h_2)}. \quad (21)$$

As usual, the respective magnetizations of the two systems are

$$m_N = \frac{1}{N_1} \sum_i^{N_1} \sigma_i, \quad (22)$$

$$n_N = \frac{1}{N_2} \sum_j^{N_2} \tau_j, \quad (23)$$

thus the Hamiltonian reads off as

$$H_N(\sigma, \tau, h) = -N_1 [\alpha_N m_N n_N + h_1 m_N + h_2 \alpha_N n_N].$$

### 3.1 The free energy in the thermodynamic limit

In order to reproduce the same scheme of the previous section, let us introduce now the  $(x, t)$ -dependent interpolating partition function

$$\begin{aligned} Z_N(x, t) &= \\ \mathbb{E}_\sigma \mathbb{E}_\tau \exp N_1 &\left( t \alpha_N m_N n_N + \frac{(\beta - t)}{2} (m_N^2 + \alpha^2 n_N^2) + x(m_N - \alpha_N n_N) + h_1 m_N + h_2 \alpha_N n_N \right) \end{aligned} \quad (24)$$

**Remark 5.** *Again we notice that the thermodynamical partition function of the model is recovered when  $t = \beta$  and  $x = 0$ .*

We can go further and define the action

$$\varphi_N(x, t) = \frac{1}{N_1} \log Z_N(x, t), \quad (25)$$

that therefore is just the pressure of the model for a suitable choice of  $(x, t)$ . Now, computing derivatives of  $\varphi_N(x, t)$ , we notice that, putting  $D_N = m_N - \alpha_N n_N$ , it is

$$\begin{aligned} \partial_t \varphi_N(x, t) &= -\frac{1}{2} \langle D_N^2 \rangle(x, t), \\ \partial_x \varphi_N(x, t) &= \langle D_N \rangle(x, t), \\ \partial_x^2 \varphi_N(x, t) &= \frac{N_1}{2} \left( \langle D_N^2 \rangle - \langle D_N \rangle^2 \right). \end{aligned}$$

The main difference with respect to the previous case is, instead, the more complicated form of the boundary condition, *i.e.* the action at  $t = 0$ . In fact we have that interactions do not factorize trivially (in a way independent by the size of the system). It is

$$\varphi_N(x, 0) = A_N^1(\beta, h_1 + x) + \alpha_N A_N^2(\alpha\beta, h_2 - x), \quad (26)$$

where  $A_N^1$  is the pressure of the Curie-Weiss model made by  $N_1$   $\sigma$  spins, and  $A_N^2$  is the same referred to the  $N_2$   $\tau$  spins. Hence, the results of the previous section give us a perfect control on the function on the r.h.s. of (26), and we have

$$\varphi_N(x, 0) = A^1(\beta, h_1 + x) + \alpha A^2(\alpha\beta, h_2 - x) + O\left(\frac{1}{N}\right). \quad (27)$$

Thus, again we can build our differential problems for the action  $\varphi_N(x, t)$

$$\begin{cases} \partial_t \varphi_N(x, t) + \frac{1}{2} (\partial_x \varphi_N(x, t))^2 + \frac{1}{2N_1} \partial_x^2 \varphi_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi_N(x, 0) = A_N^1(\beta, h_1 + x) + \alpha_N A_N^2(\alpha\beta, h_2 - x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (28)$$

and for the velocity field  $D_N(x, t)$

$$\begin{cases} \partial_t D_N(x, t) + D_N(x, t) \partial_x D_N(x, t) + \frac{1}{2N_1} \partial_x^2 D_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ D_N(x, 0) = M_N(\beta, h_1 + x) - \alpha_N N_N(\alpha\beta, h_2 - x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (29)$$

whence, as for the boundary condition for the action, we have stated

$$M_N(\beta, h_1 + x) - \alpha_N N_N(\beta, h_2 - x) = M(\beta, h_1 + x) - \alpha N(\alpha\beta, h_2 - x) + O\left(\frac{1}{\sqrt{N}}\right).$$

**Remark 6.** *We stress that our method, due to the existence of the Burger equation for the velocity field, introduces by itself the correct order parameter, without imposing it by hands. We will back on this point in the last section.*

**Remark 7.** *We have that for each collection of values  $(\beta, \alpha, h_1, h_2)$ , the function  $D_N(x, t)$  is bounded  $\forall N$ , *i.e.* the function  $\varphi_N(x, t)$  is Lipschitz continuous.*

The main difficulty here is that we have a sequence of differential problem with boundary conditions dependent by  $N$ . Anyway we can replace it with the same sequence of equation but with fixed boundary condition, that is the well defined limiting value for  $N \rightarrow \infty$  of  $\varphi_N$  and  $D_N$ . To this purpose it is useful the following

**Lemma 2.** *The two differential problems*

$$\begin{cases} \partial_t \varphi_N(x, t) + \frac{1}{2}(\partial_x \varphi_N(x, t))^2 + \frac{1}{2N_1} \partial_x^2 \varphi_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi_N(x, 0) = A^1(\beta, h_1 + x) + \alpha A^2(\alpha\beta, h_2 - x) = h_N(x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (30)$$

and

$$\begin{cases} \partial_t \bar{\varphi}_N(x, t) + \frac{1}{2}(\partial_x \bar{\varphi}_N(x, t))^2 + \frac{1}{2N_1} \partial_x^2 \bar{\varphi}_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \bar{\varphi}_N(x, 0) = A^1(\beta, h_1 + x) + \alpha A^2(\alpha\beta, h_2 - x) = h(x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (31)$$

are completely equivalent, i.e. in thermodynamic limit they have the same solution,  $\varphi_N \rightarrow \varphi$  and  $\bar{\varphi}_N \rightarrow \varphi$  and it is

$$|\varphi_N - \bar{\varphi}_N| \leq O\left(\frac{1}{N}\right).$$

*Proof.* By a Cole-Hopf transform, we can easily write the general form of  $\delta_N(x, t) = |\varphi_N(x, t) - \bar{\varphi}_N(x, t)|$  as

$$\delta_N = \frac{1}{N} \left| \log \frac{\int_{-\infty}^{+\infty} dy \Delta(y, (x, t)) e^{-NR_N(y)}}{\int_{-\infty}^{+\infty} dy \Delta(y, (x, t))} \right|,$$

where we introduced the modified heat kernel  $\Delta(y, (x, t)) = \sqrt{\frac{N}{2\pi t}} \exp(-N[(x - y)^2/2t + h(y)])$ , and  $R_N(y) = |h(y) - h_N(y)|$ , with  $\lim_N NR_N < \infty$ ,  $\forall y$ . Now we notice that because of theorem 2, it certainly exists an  $y^*$  such that

$$\sup_y R_N(y) = y^* \quad \text{and} \quad \lim_N NR_N(y^*) < \infty.$$

Hence it is

$$\begin{aligned} \delta_N(x, t) &\leq \frac{1}{N} |\log e^{-NR_N(y^*)}| \\ &= \frac{1}{N} [NR_N(y^*)] \leq O\left(\frac{1}{N}\right), \end{aligned} \quad (32)$$

that completes the proof.  $\square$

Of course a similar result holds also for the Burgers' equation for the velocity field  $D_N$ . So, finally, we must study

$$\begin{cases} \partial_t \varphi_N(x, t) + \frac{1}{2}(\partial_x \varphi_N(x, t))^2 + \frac{1}{2N_1} \partial_x^2 \varphi_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi_N(x, 0) = A^1(\beta, h_1 + x) + \alpha A^2(\alpha\beta, h_2 - x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (33)$$

and

$$\begin{cases} \partial_t D_N(x, t) + D_N(x, t) \partial_x D_N(x, t) + \frac{1}{2N_1} \partial_x^2 D_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ D_N(x, 0) = M(\beta, h_1 + x) - \alpha N(\alpha\beta, h_2 - x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \quad (34)$$

Now the path is clear, and we can state the following

**Theorem 3.** *The pressure of the generalized bipartite ferromagnet, in the thermodynamic limit, is given by:*

$$A(\beta, \alpha, h_1, h_2) = -\alpha\beta\tilde{N}\tilde{M} + \log \mathbb{E}_\sigma \exp \left[ \sigma \left( h_1 + \alpha\beta\tilde{N} \right) \right] + \alpha \log \mathbb{E}_\tau \exp \left[ \tau \left( h_2 + \beta\tilde{M} \right) \right], \quad (35)$$

where, given the well defined magnetization for the generalized CW model respectively for  $\sigma$  and  $\tau$ ,  $M(\beta, h)$  and  $N(\beta, h)$ , it is

$$\tilde{M}(\beta, \alpha, h_1, h_2) = M(\beta, h_1 - \beta M + \alpha\beta N) \quad (36)$$

$$\tilde{N}(\beta, \alpha, h_1, h_2) = N(\beta, h_2 + \beta M - \alpha\beta N). \quad (37)$$

Furthermore it is

$$|A_N(\beta, h_1, h_2) - A(\beta, \alpha, h_1, h_2)| \leq O\left(\frac{1}{N}\right). \quad (38)$$

*Proof.* Theorem 1 gives us the existence and the form of the free solution. We know [4] that the free Burger's equation can be solved along the characteristics

$$\begin{cases} t &= s \\ x &= x_0 + sD(x_0, 0), \end{cases} \quad (39)$$

where

$$D(x_0, 0) = M(\beta, h_1 + x_0) + \alpha N(\alpha\beta, h_2 - x_0),$$

and it is

$$D(x, t) = D(x_0(x, t), 0) = M(\beta, h_1 + x - tD(x_0, 0)) + \alpha N(\alpha\beta, h_2 - x + tD(x_0, 0)).$$

Then we can notice that

$$M(\beta, h_1 + x - tD(x_0, 0)) = \frac{\mathbb{E}_\sigma \sigma \exp [\sigma (h_1 + x + t\alpha N)]}{\mathbb{E}_\sigma \exp [\sigma (h_1 + x + t\alpha N)]}, \quad (40)$$

$$N(\alpha\beta, h_2 - x + tD(x_0, 0)) = \frac{\mathbb{E}_\tau \tau \exp [\tau (h_2 - x + tM)]}{\mathbb{E}_\tau \exp [\tau (h_2 - x + tM)]}, \quad (41)$$

which coincide with (36) and (37) when  $x = 0$  and  $t = \beta$ .

At this point we know that the minimum in theorem 1 is taken for  $y = x - tD(x, t)$ , and, bearing in mind the general form of the pressure of CW models, given in the last section, we have

$$\begin{aligned} [\varphi(x, t)]_{(x=0, t=\beta)} &= \left[ \frac{t}{2} D^2(x, t) - \frac{t}{2} M^2(\beta, h_1 + x - tD(x_0, 0)) - \frac{t}{2} \alpha^2 N^2(\alpha\beta, h_2 - x + tD(x_0, 0)) \right. \\ &\quad \left. + \log \mathbb{E}_\sigma \exp [\sigma (h_1 + x + t\alpha N)] + \alpha \log \mathbb{E}_\tau \exp [\tau (h_2 - x + tM)] \right]_{(x=0, t=\beta)} \\ &= A(\beta, \alpha, h_1, h_2), \end{aligned}$$

where  $A(\beta, \alpha, h_1, h_2)$  is given just by (35), bearing in mind the right definition of  $\tilde{M}$  and  $\tilde{N}$ . Now we must only prove the convergence of the true solution to the free one. But, exactly like in theorem 2, equation (38) follows by standard techniques, because of the uniform concavity of

$$\frac{(x - y)^2}{2t} + A^1(\beta, h_1 + y) + \alpha A^2(\beta, h_2 - y)$$

with respect to  $y$ , assured by theorem 1. In fact we have that, by a Cole-Hopf transform [4], the unique bounded solution of the viscous problem is

$$\varphi_N(x, t) = \frac{1}{N} \log \sqrt{\frac{N}{t}} \int \frac{dy}{\sqrt{2\pi}} \exp \left[ -N \left( \frac{(x-y)^2}{2t} + A^1(\beta, h_1 + y) + \alpha A^2(\beta, h_2 - y) \right) \right]$$

and we have, by standard estimates of a Gaussian integral, that

$$|\varphi(x, t) - \varphi_N(x, t)| \leq O\left(\frac{1}{N}\right),$$

*i.e.* also the (38) is proven.  $\square$

Finally, by this last theorem, we can easily write down the free energy of the model:

$$f(\alpha, \beta, h_1, h_2) = \alpha \tilde{N} \tilde{M} - \frac{1}{\beta} \log \mathbb{E}_\sigma \exp \left[ \sigma \left( h_1 + \alpha \beta \tilde{N} \right) \right] - \frac{\alpha}{\beta} \log \mathbb{E}_\tau \exp \left[ \tau \left( h_2 + \beta \tilde{M} \right) \right].$$

**Remark 8.** *We stress that when recovering the one party scenario (i.e.  $\alpha = 0$ ) the model trivially reduces to the well known CW in an external magnetic field, with the free energy  $-\beta f(\beta, h_1) = \ln 2 + \ln \cosh(\beta h_1)$ .*

In the last paragraph we will see how expressions like this one can be derived through a minmax principle.

### 3.2 The occurrence of a minmax principle for the free energy

As we have seen in the previous paragraph, the velocity field  $D_N(x, t)$  plays the role of order parameter for the model. Actually, in perfect analogy with other cases of interest (see for instance the last section about generalized ferromagnets, or [6]), the free energy is then obtained minimizing (or maximizing, depending on the complexity of the system, i.e. the presence of frustration [12]) the action with respect to the order parameter. In bipartite model one has two natural order parameters, *i.e.* each of which referred to the party it belongs to. From our study of bipartite ferromagnet, we know that the true order parameter is a linear combination of the two magnetizations, one for each parties,  $D = M - \alpha N$ : What is done by Lax's theorem, for example for the free energy, is taking the maximum of  $D$  on a suitable trial functional [8]

$$\begin{aligned} f(\alpha, \beta, h_1, h_2) &= \max_D \left[ -\frac{D^2}{2} + \frac{M^2}{2} + \frac{\alpha^2 N^2}{2} \right. \\ &\quad \left. - \frac{1}{\beta} \log \mathbb{E}_\sigma \exp [\sigma (h_1 + \beta M + \beta D)] - \frac{\alpha}{\beta} \log \mathbb{E}_\tau \exp [\tau (h_2 + \alpha \beta N - \beta D)] \right] \end{aligned}$$

This expression is rather unsatisfactory, since not only the order parameter of the model  $D$  appears, but even the two magnetizations  $M$  and  $N$ . Anyway we can see the model as described by two different order parameters,  $M$  and  $N$  themselves, and in the last expression one should take the extremum with respect to both  $M$  and  $N$ . Anyway we have that  $D = M - \alpha N$ , thus maximize  $D$  is equivalent to maximize  $M$  and minimize  $N$ . We must only rewrite our trial functional in terms of  $M$  and  $N$ , and we have the minmax principle for the free energy

$$f = \min_N \max_M \left[ \alpha M N - \frac{1}{\beta} \log \mathbb{E}_\sigma \exp [\sigma (h_1 + \alpha \beta N)] - \frac{\alpha}{\beta} \log \mathbb{E}_\tau \exp [\tau (h_2 + \beta M)] \right].$$

It naturally arises from the last formula that the free energy is concave with respect to  $N$  and convex with respect to  $M$ , but of course it is uniformly convex along  $M - \alpha N$ . Indeed, we have that  $M$  and  $N$  are not independent, but are related by (36) and (37), that is

$$M = \frac{\mathbb{E}_\sigma \sigma \exp [\sigma (h_1 + \alpha \beta N)]}{\mathbb{E}_\sigma \exp [\sigma (h_1 + \alpha \beta N)]}, \quad N = \frac{\mathbb{E}_\tau \tau \exp [\tau (h_2 + \beta M)]}{\mathbb{E}_\tau \exp [\tau (h_2 + \beta M)]}. \quad (42)$$

**Remark 9.** *As for the single party model, we stress that when choosing  $\mu(\sigma) = (1/2)[\delta(\sigma + 1) + \delta(\sigma - 1)]$ , i.e. dichotomic case, the self-consistent relations reduce to the already known[5]*

$$M(\beta, h_1, \alpha, N) = \tanh(h_1 + \beta \alpha N), \quad (43)$$

$$N(\beta, h_1, \alpha, N) = \tanh(h_2 + \beta M). \quad (44)$$

*However, with respect the model analyzed in [5] it should be noticed that we miss the self-contribute inside each equation (i.e.  $M \neq f(M)$  as well as  $N \neq f(N)$ ). This is ultimately due to the lacking of the self-interaction inside each party into the Hamiltonian we are considering.*

These are the true self-consistence relations of the model, analogue to (18), and we conclude that the choice of two different order parameters is redundant, since they are related. One might make the choice of putting  $N = N(M)$  and study the problem using only  $M$  as order parameter (or viceversa), but, as we have seen it is not so convenient, since a beautiful extremum principle does not seem to arise studying the system along the direction of one of the two subsystems<sup>2</sup>, i.e. along  $M$  or  $N$ . In fact we know, thanks to our technique, that the extremum is taken with respect to  $D$ .

Thus actually one has only one degree of freedom, and the minmax principle, although on one hand it gives a more satisfactory form of the flow equations, on the other hand only hides a more meaningful minimum or maximum principle. This characteristic of bipartite model seems to be quite general, and might be extended to other models of interest in future development.

## 4 Conclusion

In this paper we used a mechanical analogy, introduced and developed in [7][6], for a complete resolution of mean field ferromagnetic models with a very general class of spin r.v., i.e. with probability measure symmetric and with compact support. The free energy in the thermodynamic limit and the phase transition have appeared in our work as, respectively, the solution in the limit of vanishing viscosity of a Hamilton-Jacobi equation with diffusion, and the occurrence of a shock line for the related velocity field. Moreover, we have applied the same methods to the more interesting bipartite systems, made by two different subsystem of spins (a priori of different nature), each one interacting with the other, but with no self-interactions. We have seen that the thermodynamic limit of the pressure does exist and it is unique and we gave its explicit expression in a constructive way. Further, when introducing the Burger's equation for the velocity field, our methods automatically "choices" the proper order parameter, which turns out to be a linear combination of the magnetizations of the two subsystems, with different signs. By this property, we developed an analysis of the minmax principle, pointing out its importance relating it to the more classical min/max for the free energy (or, of course, for the pressure) for this very simple model. Noticing that the same structure can be recovered for many other models of greater interest, like bipartite spin glasses, we plan to report soon about them.

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<sup>2</sup>This is finally due to the symmetry between the  $\sigma$  subsystem and the  $\tau$  one.

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